

# Comparing Least Squares Calculations

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## Abstract

Many statistics methods require one or more least squares problems to be solved. There are several ways to perform this calculation, using objects from the base R system and using objects in the classes defined in the **Matrix** package.

We compare the speed of some of these methods on a very small example and on a example for which the model matrix is large and sparse.

## 1 Linear least squares calculations

Many statistical techniques require least squares solutions

$$\hat{\beta} = \arg \min_{\beta} \|\mathbf{y} - \mathbf{X}\beta\|^2 \quad (1)$$

where  $\mathbf{X}$  is an  $n \times p$  model matrix ( $p \leq n$ ),  $\mathbf{y}$  is  $n$ -dimensional and  $\beta$  is  $p$  dimensional. Most statistics texts state that the solution to (1) is

$$\hat{\beta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \quad (2)$$

when  $\mathbf{X}$  has full column rank (i.e. the columns of  $\mathbf{X}$  are linearly independent) and all too frequently it is calculated in exactly this way.

### 1.1 A small example

As an example, let's create a model matrix, **mm**, and corresponding response vector, **y**, for a simple linear regression model using the **Formaldehyde** data.

```
> data(Formaldehyde)
+
> str(Formaldehyde)

'data.frame':      6 obs. of  2 variables:
 $ carb   : num   0.1 0.3 0.5 0.6 0.7 0.9
 $ optden: num   0.086 0.269 0.446 0.538 0.626 0.782
```

```
> (m <- cbind(1, Formaldehyde$carb))
```

```
      [,1] [,2]
[1,]    1 0.1
[2,]    1 0.3
[3,]    1 0.5
[4,]    1 0.6
[5,]    1 0.7
[6,]    1 0.9
```

```
> (yo <- Formaldehyde$optden)
```

```
[1] 0.086 0.269 0.446 0.538 0.626 0.782
```

Using `t` to evaluate the transpose, `solve` to take an inverse, and the `%%` operator for matrix multiplication, we can translate 2 into the S language as

```
> solve(t(m) %% m) %% t(m) %% yo
+
```

```
      [,1]
[1,] 0.005085714
[2,] 0.876285714
```

On modern computers this calculation is performed so quickly that it cannot be timed accurately in R <sup>1</sup>

```
> system.time(solve(t(m) %% m) %% t(m) %% yo)
+
```

```
      user  system elapsed
0.001    0.000    0.000
```

and it provides essentially the same results as the standard `lm.fit` function that is called by `lm`.

```
> dput(c(solve(t(m) %% m) %% t(m) %% yo))
+
```

```
c(0.00508571428571428, 0.876285714285715)
```

```
> dput(unname(lm.fit(m, yo)$coefficients))
```

```
c(0.00508571428571408, 0.876285714285715)
```

---

<sup>1</sup>From R version 2.2.0, `system.time()` has default argument `gcFirst = TRUE` which is assumed and relevant for all subsequent timings

## 1.2 A large example

For a large, ill-conditioned least squares problem, such as that described in Koenker and Ng (2003), the literal translation of (2) does not perform well.

```
> library(Matrix)
+
> data(KNex, package = "Matrix")
> y <- KNex$y
> mm <- as(KNex$mm, "matrix") # full traditional matrix
> dim(mm)

[1] 1850  712

> system.time(naive.sol <- solve(t(mm) %*% mm) %*% t(mm) %*% y)

      user system elapsed 
4.164    0.029    4.200
```

Because the calculation of a “cross-product” matrix, such as  $\mathbf{X}^\top \mathbf{X}$  or  $\mathbf{X}^\top \mathbf{y}$ , is a common operation in statistics, the `crossprod` function has been provided to do this efficiently. In the single argument form `crossprod(mm)` calculates  $\mathbf{X}^\top \mathbf{X}$ , taking advantage of the symmetry of the product. That is, instead of calculating the  $712^2 = 506944$  elements of  $\mathbf{X}^\top \mathbf{X}$  separately, it only calculates the  $(712 \cdot 713)/2 = 253828$  elements in the upper triangle and replicates them in the lower triangle. Furthermore, there is no need to calculate the inverse of a matrix explicitly when solving a linear system of equations. When the two argument form of the `solve` function is used the linear system

$$(\mathbf{X}^\top \mathbf{X}) \hat{\boldsymbol{\beta}} = \mathbf{X}^\top \mathbf{y} \quad (3)$$

is solved directly.

Combining these optimizations we obtain

```
> system.time(cpod.sol <- solve(crossprod(mm), crossprod(mm,y)))
+

      user system elapsed 
1.112    0.008    1.122

> all.equal(naive.sol, cpod.sol)

[1] TRUE
```

On this computer (2.0 GHz Pentium-4, 1 GB Memory, Goto’s BLAS, in Spring 2004) the `crossprod` form of the calculation is about four times as fast as the naive calculation. In fact, the entire `crossprod` solution is faster than simply calculating  $\mathbf{X}^\top \mathbf{X}$  the naive way.

```
> system.time(t(mm) %*% mm)
+
```

```
      user  system elapsed
1.998    0.001    2.007
```

Note that in newer versions of R and the BLAS library (as of summer 2007), R's `%*%` is able to detect the many zeros in `mm` and shortcut many operations, and is hence much faster for such a sparse matrix than `crossprod` which currently does not make use of such optimizations. This is not the case when R is linked against an optimized BLAS library such as GOTO or ATLAS. Also, for fully dense matrices, `crossprod()` indeed remains faster (by a factor of two, typically) independently of the BLAS library:

```
> fm <- mm
+
> set.seed(11)
> fm[] <- rnorm(length(fm))
> system.time(c1 <- t(fm) %*% fm)
```

```
      user  system elapsed
1.985    0.003    1.990
```

```
> system.time(c2 <- crossprod(fm))
```

```
      user  system elapsed
0.909    0.000    0.910
```

```
> stopifnot(all.equal(c1, c2, tol = 1e-12))
```

### 1.3 Least squares calculations with Matrix classes

The `crossprod` function applied to a single matrix takes advantage of symmetry when calculating the product but does not retain the information that the product is symmetric (and positive semidefinite). As a result the solution of (3) is performed using general linear system solver based on an LU decomposition when it would be faster, and more stable numerically, to use a Cholesky decomposition. The Cholesky decomposition could be used but it is rather awkward

```
> system.time(ch <- chol(crossprod(mm)))
+
      user  system elapsed
1.003    0.000    1.004

> system.time(chol.sol <-
+             backsolve(ch, forwardsolve(ch, crossprod(mm, y),
+             upper = TRUE, trans = TRUE)))
```

```

      user  system elapsed
0.012    0.000    0.012

> stopifnot(all.equal(chol.sol, naive.sol))

The Matrix package uses the S4 class system (Chambers, 1998) to retain
information on the structure of matrices from the intermediate calculations.
A general matrix in dense storage, created by the Matrix function, has class
"dgeMatrix" but its cross-product has class "dpoMatrix". The solve methods
for the "dpoMatrix" class use the Cholesky decomposition.

> mm <- as(KNex$mm, "dgeMatrix")
+
> class(crossprod(mm))

[1] "dpoMatrix"
attr(,"package")
[1] "Matrix"

> system.time(Mat.sol <- solve(crossprod(mm), crossprod(mm, y)))

      user  system elapsed
1.333    0.001    1.335

> stopifnot(all.equal(naive.sol, unname(as(Mat.sol, "matrix"))))

Furthermore, any method that calculates a decomposition or factorization
stores the resulting factorization with the original object so that it can be reused
without recalculation.

> xpx <- crossprod(mm)
+
> xpy <- crossprod(mm, y)
> system.time(solve(xpx, xpy))

      user  system elapsed
0.099    0.000    0.099

> system.time(solve(xpx, xpy)) # reusing factorization

      user  system elapsed
0.001    0.000    0.001

The model matrix mm is sparse; that is, most of the elements of mm are zero.
The Matrix package incorporates special methods for sparse matrices, which
produce the fastest results of all.

> mm <- KNex$mm
+
> class(mm)

```

```

[1] "dgCMatrix"
attr(,"package")
[1] "Matrix"

> system.time(sparse.sol <- solve(crossprod(mm), crossprod(mm, y)))

   user   system elapsed 
0.005   0.001   0.006 

> stopifnot(all.equal(naive.sol, unname(as(sparse.sol, "matrix"))))

```

As with other classes in the `Matrix` package, the `dsCMatrix` retains any factorization that has been calculated although, in this case, the decomposition is so fast that it is difficult to determine the difference in the solution times.

```

> xpx <- crossprod(mm)
+
> xpy <- crossprod(mm, y)
> system.time(solve(xpx, xpy))

   user   system elapsed 
0.002   0.000   0.001 

> system.time(solve(xpx, xpy))

   user   system elapsed 
0.001   0.000   0.001 

```

## Session Info

```

> toLatex(sessionInfo())
+
  • R version 2.13.0 beta (2011-03-31 r55221), x86_64-unknown-linux-gnu
  • Locale: LC_CTYPE=de_CH.UTF-8, LC_NUMERIC=C, LC_TIME=en_US.UTF-8,
    LC_COLLATE=C, LC_MONETARY=C, LC_MESSAGES=de_CH.UTF-8,
    LC_PAPER=de_CH.UTF-8, LC_NAME=C, LC_ADDRESS=C, LC_TELEPHONE=C,
    LC_MEASUREMENT=de_CH.UTF-8, LC_IDENTIFICATION=C
  • Base packages: base, datasets, grDevices, graphics, methods, stats, utils
  • Other packages: Matrix 0.999375-49, lattice 0.19-17
  • Loaded via a namespace (and not attached): grid 2.13.0, tools 2.13.0

> if(identical(1L, grep("linux", R.version[["os"]]))) { ## Linux - only ---
+   Scpu <- sfsmisc::Sys.procinfo("/proc/cpuinfo")
+   Smem <- sfsmisc::Sys.procinfo("/proc/meminfo")
+   print(Scpu[c("model name", "cpu MHz", "cache size", "bogomips")])
+   print(Smem[c("MemTotal", "SwapTotal")])
+ }

```

```
model name AMD Phenom(tm) II X4 925 Processor
cpu MHz 800.000
cache size 512 KB
bogomips 5600.21

MemTotal 7936268 kB
SwapTotal 16777212 kB
```

## References

- John M. Chambers. *Programming with Data*. Springer, New York, 1998. ISBN 0-387-98503-4.
- Roger Koenker and Pin Ng. SparseM: A sparse matrix package for R. *J. of Statistical Software*, 8(6), 2003.