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1 Release 4.3 Updates.

As with the previous Rel. 4 update manual, this represents new changes reflected in the third subrelease. The main reason for not combining this material with updates in the previous report in one document are the changes in typesetting style and the delays that would be caused with changing style in the previous report.

Changes most obvious to users of the program proj are the addition of new projections—the total is now about 110. For programmers using the projection library, the main change is in how to limit the list of projections linked into application programs. Additional, internal changes were made to ease maintenance of the system, but they should be transparent to both user and programmer.

Manual Style. This update is concerned with only documenting projections. Waffling by the author about what should be included or ignored are beginning to converge to the style presented here. Description of the Pseudocylindrical class of projections that follows is nearly complete and will probably not change greatly in the final documentation. A few of previous Miscellaneous projections and new additions are included as well as a section on the General Oblique projection.

It was also decided to include the formulary as part of documentation for reference by the serious reader and to make an explicit definition of what is considered by the author to be the mathematical definition of each projection in this system.

Any comments as to this new style are appreciated.

Apologies. Because automatic typesetting programs do not always make the best choices, there are several undesirable locating of figures relative to text. These can usually be overcome by extra effort by the author, but such manipulations are likely to be destroyed by later, overall document alterations. Thus, little effort was expended at this preliminary stage in “beautifying” the text.
2 Pseudocylindrical Projections.

Pseudocylindrical projections are a result of efforts to minimize the distortion of the polar regions of the cylindrical projections by bending the meridians toward the center of the map as a function of longitude while maintaining the cylindrical characteristic of parallel parallels. These projections are almost exclusively used for small scale global displays and, except for the Sinusoidal projection, only derived for a spherical Earth. Because of the basic definition of pseudocylindrical projections, none are conformal, but many are equal area.

To further reduce distortion, pseudocylindrical are often presented in interrupted form that are made by joining several regions with appropriate central meridians and false easting and clipping boundaries. Figs. 1 and 2 show typical construction that are suited for showing respective global land and oceanic regions. To reduce the lateral size of the map, some uses remove an irregular, North-South strip of the mid-Atlantic region so that the western tip of Africa is plotted north of the eastern tip of South America.

Pseudocylindrical are sub-classed into groups based upon the shape of the meridians: sinusoidal, elliptical, parabolic, hyperbolic, rectilinear and miscellaneous. An additional category is based upon whether the meridians come to a point at the pole or are terminated along a straight line—flat-topped.

2.0.1 Computations.

A complicating factor in computing the forward projection for pseudocylindricals is that some of the projection formulae use a parametric variable, typically \( \theta \), which is a function of \( \phi \). In some cases, the parametric equation is not directly solvable for \( \theta \) and requires use of Newton-Raphson’s method of iterative finding the root of \( P(\theta) \). The defining equations for these cases are thus given in the form of \( P(\theta) \) and its derivative, \( P'(\theta) \), and an estimating initial value for \( \theta_0 = f(\phi) \). Refinement of \( \theta \) is made by

\[
\theta \leftarrow \theta - \frac{P(\theta)}{P'(\theta)}
\]

until \( |P(\theta)/P'(\theta)| \) is less than predefined tolerance.

When known, formula constant factors are given in rational form (e.g. \( \sqrt{2}/2 \)) rather than a decimal value (0.7071) so that the precision used in the resultant program code constants is determined by the programmer. However, source material may only provide decimal values, typically to 5 or 6 decimal digits. This is adequate in most cases, but has caused problems with the convergence of a Newton-Raphson determination and degrades the determination of numerical derivatives.

Because several of the pseudocylindrical projections have a common computational base, they are grouped into a single module with multiple initializing entry points. This may lead to a minor loss of efficiency, such as adding a zero term in the simple Sinusoidal case of the the Generalized Sinusoidal (2.1.1).

2.0.2 Sources.

The principle source for pseudocylindrical formulae is [7]. Many formulae are repeated in Snyder’s later works [11] and [10], with the latter adding a few additional projections. Mahling, [2], covers several of the Russian projections but the formulae are often difficult to read. Mahling also has given fourteen pseudocylindrical formulae in [3, Appendix 1] but some discrepancies are found when compared to Snyder’s work. For the Robinson Projection (2.6.6), [6] was consulted to verify precision of tabular values and lack of specification of interpolation method. Common pseudocylindricals formulae are also found in Pearson’s work: [4] and [5]. Ellipsoid formulae for the Sinusoidal projection is from [9].

2.1 Sinusoidal Pseudocylindricals

2.1.1 Generalized Sinusoidal

McBryde and Thomas developed a generalized formulas for several of the pseudocylindricals with sinusoidal meridians:

\[
\begin{align*}
x &= C\lambda(m + \cos \theta)/(m + 1) \\
y &= C\theta
\end{align*}
\]
### Table 1: List of pseudocylindrical projections

<table>
<thead>
<tr>
<th>Projection name</th>
<th>Fig.</th>
<th>Class</th>
<th>Sect.</th>
<th>$H/V$</th>
<th>$P/H$</th>
<th>+proj=</th>
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</table>
2.1 Sinusoidal Pseudocylindricals

\[ C = \frac{\sqrt{(m+1)}}{n} \]
\[ P(\theta) = m\theta + \sin \theta - n \sin \phi \]
\[ P'(\theta) = m + \cos \theta \]
\[ \theta_0 = \phi \]

<table>
<thead>
<tr>
<th>Sinusoidal (Sanson-Flamsteed)</th>
<th>( m )</th>
<th>( n )</th>
<th>( C )</th>
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<td>( 1 + \pi/2 )</td>
<td>( 2\sqrt{2 + \pi} )</td>
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<td>McBryde-Thomas Flat-Polar Sinusoidal</td>
<td>( 1/2 )</td>
<td>( 1 + \pi/4 )</td>
<td>( \sqrt{6/(4 + \pi)} )</td>
</tr>
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</table>

Parameters \( n=n \) and \( m=n \) are required for the general form, \texttt{proj=gn_sinu}. The projection is equal-area for all cases.

When \( m = 0 \), \( P(\theta) \) simplifies and does not need Newton-Raphson iterative solution and in the Sinusoidal case, \( \theta = \phi \).

**Elliptical Earth.** The Sinusoidal projection for the ellipsoidal case becomes:

\[ x = \lambda \cos \phi (1 - e^2 \sin^2 \phi)^{-1/2} \]
\[ y = M(\phi) \]

The inverse is readily solved by determining \( \phi \) from \( M^{-1}(y) \) and substituting into the \( x \) equation for the solution of \( \lambda \).

2.1.2 Urmaev Flat-Polar Sinusoidal Series

This equal-area system is similar to 2.1.1 where the respective \( x \) and \( y \) axis are multiplied and divided by \( \sqrt{2/3} \) and where \( m = 0 \). The parameter, \( n=n \), must be specified and is restricted by \( 0 < n \leq 1 \). The Wagner I (Kavraisky VI) projection is generated when \( n=\sqrt{3}/2 \) or by selecting \texttt{proj=wagi}.

\[ x = m\lambda \cos \theta \]
\[ y = 3\theta/mn \]
\[ \sin \theta = n \sin \phi \]

where \( m = 2\sqrt{3}/3 \). Latitude of true scale on the central meridian is determined by the relation: \( \sin^2 \phi_{ts} = (9 - 4\sqrt{3})/(9 - 4n^2\sqrt{3}) \). The ratio of the length of the poles to the equator is determined by \( \sqrt{1-n^2} \).

2.1.3 Eckert V

\[ x = \lambda(1 + \cos \phi)/\sqrt{2 + \pi} \]
\[ y = 2\phi/\sqrt{2 + \pi} \]

2.1.4 Winkel I

Option \texttt{lat_ts=\phi_{ts}} establishes latitude of true scale on central meridian (default = 0° and thus the same as Eckert V). Not equal-area but if \( \cos \phi_{ts} = 2/\pi \) (\texttt{lat_ts=50d28'}) the total area of the global map is correct. If \( \phi_{ts} = 0 \)

\[ x = \lambda(\cos \phi_{ts} + \cos \phi)/2 \]
\[ y = \phi \]
2.1.5 Wagner III

\[
x = \frac{\cos \phi_t}{\cos(\frac{2\phi_t}{3})} \lambda \cos(\frac{2\phi}{3}) \\
y = \phi
\]

2.1.6 Wagner II

\[
x = 0.92483 \lambda \cos \theta \\
y = 1.38725 \theta \\
\sin \theta = 0.88022 \sin(0.8855\phi)
\]

2.1.7 Foucaut Sinusoidal.

The y-axis is based upon a weighted mean of the cylindrical equal-area and the sinusoidal projections. Parameter $n$ is the weighting factor where $0 \leq n \leq 1$.

\[
x = \lambda \cos \phi/(n + (1 - n) \cos \phi) \\
y = n\phi + (1 - n) \sin \phi
\]

For the inverse, the Newton-Raphson method can be used to determine $\phi$ from the equation for $y$ above. As $n \to 0$ and $\phi \to \pi/2$, convergence is slow but for $n = 0$, $\phi = \sin^{-1} y$. 
2.2 Elliptical Pseudocylindricals.

2.2.1 Mollweide, Wagner IV (Putninš P₂), and Wagner V

Mollweide and Wagner IV are equal area, but Wagner V is not.

\[
x = C_x \lambda \cos(\theta/2) \\
y = C_y \sin(\theta/2) \\
C_x = 0.90977 \quad \text{for Wagner V} \\
C_y = 1.65014 \quad \text{for Wagner V} \\
\theta_0 = \phi \\
r = \sqrt{2\pi \sin p/(2p + \sin 2p)}
\]

and where \( p = \pi/2 \) for Mollweide and \( p = \pi/3 \) for Wagner IV. The parametric equation converges slowly for the Mollweide case.

2.2.2 Eckert IV

\[
x = 2\lambda(1 + \cos \theta)/\sqrt{\pi(4 + \pi)} \\
y = 2\sqrt{\pi/(4 + \pi)} \sin \theta \\
P(\theta) = \theta + \sin 2\theta + 2 \sin \theta - \frac{(4 + \pi)}{2} \sin \phi \\
= \theta + \sin \theta(\cos \theta + 2) - \frac{(4 + \pi)}{2} \sin \phi \\
P'(\theta) = 2 + 4 \cos 2\theta + 4 \cos \theta \\
= 1 + \cos \theta(\cos \theta + 2) = 1 + \sin^2 \theta \\
\theta_0 = 0.895168 \phi + \phi \cos \theta - \sin^2 \theta \\
\phi = 0.895168 \phi + 0.0218849 \phi^3 + 0.00826809 \phi^5
\]

The parametric equation converges slowly as \( \phi \) nears \( \pi/2 \) and \( \theta \) approaches \( \pi/3 \).

2.2.3 Putninš P₂

\[
x = 1.89490\lambda(\cos \theta - 1/2) \\
y = 1.71848 \sin \theta \\
P(\theta) = 2\theta + \sin 2\theta - 2 \sin \theta - [(4\pi - 3\sqrt{3})/6] \sin \phi \\
= \theta + \sin \theta(\cos \theta - 1) - [(4\pi - 3\sqrt{3})/12] \sin \phi \\
P'(\theta) = 2 + 2 \cos 2\theta + 2 \cos \theta \\
= 1 + \cos \theta(\cos \theta - 1) - \sin^2 \theta \\
\theta_0 = 0.615709 \phi + 0.0090953 \phi^3 + 0.0046292 \phi^5
\]

The parametric equation converges slowly as \( \phi \) nears \( \pi/2 \) and \( \theta \) approaches \( \pi/3 \).

2.2.4 Hatano

\[
x = 0.85\lambda \cos \theta \\
y = C_y \sin \theta
\]
2 PSEUDOCYLINDRICAL PROJECTIONS.

\[ P(\theta) = 2\theta + \sin 2\theta - C_p \sin \phi \]
\[ P'(\theta) = 2(1 + \cos 2\theta) \]
\[ \theta_0 = 2\phi \]

\[ \begin{array}{c|cc}
\phi > 0 & C_y & C_p \\
\hline
1.75859 & 2.67595 \\
\phi < 0 & 1.93052 & 2.43763
\end{array} \]

For \( \phi = 0 \), \( y \leftarrow 0 \) and \( x \leftarrow 0.85\lambda \).

2.2.5 Eckert III, Putniniš P_1, Wagner VI (Putniniš P'_1), and Kavraisky VII

None of these projections are equal-area and are flat-polar when coefficient \( A \neq 0 \).

\[ x = C_x \lambda (A + \sqrt{1 - B(\phi/\pi)^2}) \]

\[ y = C_y \phi \]

<table>
<thead>
<tr>
<th></th>
<th>( C_x )</th>
<th>( C_y )</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Putniniš P_1</td>
<td>0.94745</td>
<td>0.94745</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Wagner VI</td>
<td>1.89490</td>
<td>0.94745</td>
<td>-1/2</td>
<td>3</td>
</tr>
<tr>
<td>Eckert III</td>
<td>( \sqrt{\pi(4+\pi)}^2 )</td>
<td>( 4/\sqrt{\pi(4+\pi)} )</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Kavraisky VII</td>
<td>( \sqrt{3}/2 )</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

2.3 Hyperbolic Pseudocylindricals

In this group where the meridians are hyperbolic only four Putniniš forms are given.

2.3.1 Putniniš P_6 and P'_6

Putniniš P_6 and P'_6 projections are equal-area with respective pointed and flat poles defined by:

\[ x = C_x \lambda (D - (1 + p^2)^{1/2}) \]
\[ y = C_yp \]
\[ P(p) = (A - (1 + p^2)^{1/2})p - \ln(p + (1 + p^2)^{1/2}) - B \sin \phi \]
\[ P'(p) = A - 2\sqrt{1 + p^2} \]
\[ p_0 = \phi \]

where
2.4 Parabolic Pseudocylindricals

2.4.1 Craster (Putnînş P5)

A pointed pole, equal-area projection with standard parallels at 36°46'.

\[
\begin{align*}
x &= \sqrt{3/\pi} \lambda [2 \cos(2\phi/3) - 1] \\
y &= \sqrt{3\pi} \sin(\phi/3)
\end{align*}
\]

2.4.2 Putnînş P5' and Werenskiold I

This is the flat pole version of Putnînş’s P5 or Craster’s Parabolic:

\[
\begin{align*}
x &= C_x \lambda \cos \theta / \cos(\theta/3) \\
y &= C_y \sin(\theta/3) \\
\sin \theta &= (5\sqrt{2}/8) \sin \phi
\end{align*}
\]

where

\[
\begin{array}{c|c|c}
P_5' & \text{Werens. I} \\
\hline
C_x & 2\sqrt{0.6/\pi} & 1.0 \\
C_y & 2\sqrt{1.2/\pi} & \pi\sqrt{2}
\end{array}
\]

2.3.2 Putnînş P6 and P6'

Putnînş P6 and P6' projections have equally spaced parallels and respectively pointed and flat poles:

\[
\begin{align*}
x &= 1.01346\lambda(A - B\sqrt{1 + 12\phi^2/\pi^2}) \\
y &= 1.01346\phi
\end{align*}
\]

<table>
<thead>
<tr>
<th>P6</th>
<th>P6'</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_x)</td>
<td>1.01346</td>
</tr>
<tr>
<td>(D)</td>
<td>2</td>
</tr>
<tr>
<td>(C_y)</td>
<td>0.91910</td>
</tr>
<tr>
<td>(A)</td>
<td>4.00000</td>
</tr>
<tr>
<td>(B)</td>
<td>2.14714</td>
</tr>
</tbody>
</table>

Figure 20: Putnînş P6.

Figure 21: Putnînş P6'.

Figure 22: Putnînş P5.

Figure 23: Putnînş P5'.

Figure 24: Craster.
2.4.3 Putnišiš P₃ and P₃′

\[ x = \sqrt{\frac{2}{\pi}} \lambda (1 - A \phi^2 / \pi^2) \]
\[ y = \sqrt{\frac{2}{\pi}} \phi \]

where \( A \) is 4 and 2 for respective \( P₃ \) and \( P₃' \).

2.4.4 McBryde-Thomas Flat-Polar Parabolic

\[ x = \sqrt{\frac{6}{\pi}} \frac{7}{3} \lambda [1 + 2 \cos \theta / \cos(\theta/3)] \]
\[ y = 3 \sqrt{\frac{6}{\pi}} \sin(\theta/3) \]
\[ P(\theta) = 1.125 \sin(\theta/3) - \sin^3(\theta/3) - 0.4375 \sin \phi \]
\[ P'(\theta) = [0.375 - \sin^2(\theta/3)] \cos(\theta/3) \]
\[ \theta_0 = \phi \]
2.6 Miscellaneous pseudo/Pseudocylindricals.

2.6.1 Sine-Tangent Series

Sine series:
\[
x = \left(\frac{q}{p}\right)\lambda \cos \phi / \cos(\phi/q) \\
y = p \sin(\phi/q)
\]

Tangent series:
\[
x = \left(\frac{q}{p}\right)\lambda \cos \phi \cos^2(\phi/q) \\
y = p \tan(\phi/q)
\]

2.6.2 McBryde-Thomas Flat-Polar Sine (No. 1).

\[
x = 0.22248\lambda[1 + 3 \cos \theta / \cos(\theta/1.36509)] \\
y = 1.44492 \sin(\theta/1.36509) \\
P(\theta) = 0.45503 \sin(\theta/1.36509) + \sin \theta - 1.41546 \sin \phi
\]
2.6.3 McBryde-Thomas Flat-Polar Quartic

\[ x = \lambda (1 + 2 \cos \theta / \cos(\theta/2))[3\sqrt{2} + 6]^{-1/2} \]
\[ y = (2\sqrt{3}\sin(\theta/2)[2 + \sqrt{2}]^{-1/2} \]
\[ P(\theta) = \sin(\theta/2) + \sin \theta - (1 + \sqrt{2}/2) \sin \phi \]
\[ P'(\theta) = (1/2) \cos(\theta/2) + \cos \theta \]
\[ \theta = \phi \]

2.6.4 Boggs Eumorphic

\[ x = 2.00276\lambda (\sec \phi + 1.11072 \sec \theta) \]
\[ y = 0.49931(\phi + \sqrt{2}\sin \theta) \]
\[ P(\theta) = 2\theta + \sin 2\theta - \pi \sin \phi \]
\[ P'(\theta) = 2 + 2 \cos 2\theta \]
\[ \theta = \phi \]

2.6.5 Nell-Hammer

\[ x = \lambda (1 + \cos \phi)/2 \]
\[ y = 2(\phi - \tan(\phi/2)) \]
2.6 Miscellaneous pseudo/Pseudocylindricals.

<table>
<thead>
<tr>
<th>$\phi^\circ$</th>
<th>Y</th>
<th>X</th>
<th>$\phi^\circ$</th>
<th>Y</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000</td>
<td>1.0000</td>
<td>50</td>
<td>0.6176</td>
<td>0.8679</td>
</tr>
<tr>
<td>5</td>
<td>0.0620</td>
<td>0.9986</td>
<td>55</td>
<td>0.6769</td>
<td>0.8350</td>
</tr>
<tr>
<td>10</td>
<td>0.1240</td>
<td>0.9954</td>
<td>60</td>
<td>0.7346</td>
<td>0.7986</td>
</tr>
<tr>
<td>15</td>
<td>0.1860</td>
<td>0.9900</td>
<td>65</td>
<td>0.7903</td>
<td>0.7597</td>
</tr>
<tr>
<td>20</td>
<td>0.2480</td>
<td>0.9822</td>
<td>70</td>
<td>0.8435</td>
<td>0.7186</td>
</tr>
<tr>
<td>25</td>
<td>0.3100</td>
<td>0.9730</td>
<td>75</td>
<td>0.8936</td>
<td>0.6732</td>
</tr>
<tr>
<td>30</td>
<td>0.3720</td>
<td>0.9600</td>
<td>80</td>
<td>0.9394</td>
<td>0.6213</td>
</tr>
<tr>
<td>35</td>
<td>0.4340</td>
<td>0.9427</td>
<td>85</td>
<td>0.9761</td>
<td>0.5722</td>
</tr>
<tr>
<td>40</td>
<td>0.4968</td>
<td>0.9216</td>
<td>90</td>
<td>1.0000</td>
<td>0.5322</td>
</tr>
<tr>
<td>45</td>
<td>0.5571</td>
<td>0.8962</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Robinson did not define how intermediate values were to be interpolated between the 5$^\circ$ intervals. The proj system uses a set of bicubic splines determined for each $X$–$Y$ set with zero second derivatives at the poles. GCTP [12, program comments] uses Stirling’s interpolation with second differences.

2.6.7 Denoyer

\[
\begin{align*}
  x &= \lambda \cos[(0.95 - \lambda/12 + \lambda^3/600)\phi] \\
  y &= \phi
\end{align*}
\]

2.6.8 Fahey

\[
\begin{align*}
  x &= \lambda \cos 35^\circ \sqrt{1 - \tan^2(\phi/2)} \\
  y &= (1 + \cos 35^\circ) \tan(\phi/2)
\end{align*}
\]

2.6.9 Ginsburg VIII or TsNIIGAiK

\[
\begin{align*}
  x &= \lambda(1 - 0.162388\phi^2)(0.87 - 0.000952426\lambda^4) \\
  y &= \phi(1 + \phi^3/12)
\end{align*}
\]

2.6.10 Loximuthal

All straight lines radiating from the point where $\text{lat}_1=\phi_1$ intersects the central meridian are loxodromes (rhumb lines) and scale along the loxodomes.

Figure 42: Denoyer.

Figure 43: Fahey.

Figure 44: Ginsburg VIII.

Figure 45: Loximuthal. $\text{lat}_1=51d28$, Greenwich, England.
is true.

\[
\begin{align*}
x &= \frac{\lambda(\phi - \phi_1)}{\ln \tan(\pi/4 + \phi_1/2)} - \\
&\quad \ln \tan(\pi/4 + \phi_1/2) \text{ for } \phi \neq \phi_1 \\
y &= \frac{\lambda \cos \phi_1}{2} \text{ for } \phi = \phi_1
\end{align*}
\]

2.6.11 Winkel II

Arithmetic mean of Equirectangular and Mollweide and is not equal-area. Parameter \( \text{lat}_1 = \phi_1 \) controls standard parallel and width of flat polar extent.

\[
\begin{align*}
x &= \frac{\lambda(\cos \theta + \cos \phi_1)}{2} \\
y &= \frac{\pi(\sin \theta + 2\phi_1/\pi)}{4} \\
P(\theta) &= 2\theta + \sin 2\theta - \pi \sin \phi \\
P'(\theta) &= 2 + 2\cos 2\theta \\
\theta_0 &= 0.9\phi
\end{align*}
\]

As with Mollweide, \( P \) converges slowly as \( \phi \to \pi/2 \) and \( \theta \to \pi/2 \).

2.6.12 Urmaev V Series

\[
\begin{align*}
x &= m\lambda \cos \theta \\
y &= \frac{\theta(1 + q\theta^2/3)}{(mn)} \\
\sin \theta &= n \sin \phi
\end{align*}
\]

where \( m = 2\sqrt{3}/3, n = 0.8 \) and \( q = 0.414524 \) are default values that have been employed in some atlases.

2.6.13 Goode Homolosine

This projection is a combination of the Sinusoidal and Mollweide projections where the Sinusoidal is used for the equitorial regions between the latitudes of \( \pm 40^\circ 44' \) and a corrected Mollweide projection used for the remaining polar regions. The Mollweide correction is to the \( y \) axis with 0.05280 subtracted for northern latitudes and added for southern latitudes. Most often used in the interrupted form (Figs. 1 and 2).
3 Miscellaneous Projections.

Projections that do not clearly fall into previous classifications are placed into the miscellaneous class. This class is further subdivided into subgroupings that are based upon general appearance rather than inherent mathematical or derivative properties.

3.1 Near Pseudocylindricals.

This group of projections are similar to the pseudocylindrical class but with the major exception that they have curved parallels.

3.1.1 Aitoff

\[ x = 2\theta \cos \phi \sin(\lambda/2)/\sin \theta \]
\[ y = \theta \sin \phi/\sin \theta \]
\[ \cos \theta = \cos \phi \cos(\lambda/2) \]

If \( \lambda = \phi = 0 \), then \( x = y = 0 \).

3.1.2 Winkel Tripel

Winkel Tripel is the arithmetic mean of the Aitoff and Equidistant Cylindrical projections with the latter’s \( \phi_{ts} \) (latitude of true scale) becoming \( \phi_1 \). If \( \text{lat}_1=\phi_1 \) is not specified, Winkel’s value of \( \phi_1 = \cos^{-1}(2/\pi) \) or \( 50^\circ 27'35.1945'' \) is used. For Bartholomew’s variant, use \( \text{lat}_1=40 \).

3.1.3 Hammer (Hammer-Aitoff) and Eckert-Greifendorff.

A popular alternative to pseudocylindricals.

\[ x = (\sqrt{2}MD) \cos \phi \sin(W\lambda) \]
\[ y = (\sqrt{2}D/M) \sin \phi \]
\[ D = \sqrt{1 + \cos \phi \sin(W\lambda)} \]

where \( W = 0.5 \) for Hammer and \( W = 0.25 \) for Eckert-Greifendorff. \( M = 1 \) unless overridden with \( \text{M= option} \).
3 MISCELLANEOUS PROJECTIONS.

3.1.6 Laskowski.

\[
\begin{align*}
\sin \theta &= \sin 65^\circ \sin \phi \\
\cos \alpha &= \cos \theta \cos(\lambda/3)
\end{align*}
\]

\[
\begin{align*}
x &= \sum_{i=0} \sum_{j=0} a_{ij} \lambda^i \phi^j \\
y &= \sum_{i=0} \sum_{j=0} b_{ij} \lambda^i \phi^j
\end{align*}
\]

where non-zero coefficients are:

\[
\begin{align*}
a_{10} &= 0.975534 \\
a_{12} &= -0.119161 \\
a_{32} &= -0.0143059 \\
a_{14} &= -0.0547009 \\
b_{01} &= 1.00384 \\
b_{21} &= 0.0802894 \\
b_{03} &= 0.0998909 \\
b_{41} &= 0.000199025 \\
b_{23} &= -0.0285500 \\
b_{05} &= -0.0491032
\end{align*}
\]
4 Creating Oblique Projections.

All of the spherical forms of the projections in the `proj` system can be transformed into an _oblique aspect_ by making an axis transformation of the geographic coordinates with the following formula:

\[
\begin{align*}
\phi' &= \sin^{-1}(\sin \phi_p \sin \phi - \cos \phi_p \cos \phi \cos \lambda) \\
\lambda' &= \lambda + \frac{\tan(\cos \phi \sin \lambda, \sin \phi_p \cos \phi \cos \lambda + \cos \phi_p \sin \phi)}{\sin \phi_p \cos \phi \cos \lambda + \cos \phi_p \sin \phi}
\end{align*}
\]

where \(\lambda_p\) and \(\phi_p\) are the coordinates of the North pole of the transformed coordinate system on the original coordinate system. To use this transformation, the `+o_proj=name` parameter is used where _name_ is the acronym of one of the standard projections—`+o_proj` is used instead of `+proj`. Parameters `+o_lat=\phi_p` and `+o_lon=\lambda_p` are used to set the translated pole position. Any other parameters related to the selected projection _name_ are entered as otherwise documented. The parameter `lon_0` used to shift the central meridian is applied before the transformation in `+ob_tran` so the effect is to rotate the meridians about the transformed pole and not the pole of the target projection.

To illustrate this procedure, the National Geographic Societies’ _Atlas of the World_ [1, p. 4] uses the Oblique McBryde-Thomas Flat-Polar projection for a shaded-relief map of the world. Unfortunately, they do not fully annotate the figure (see [8] for comments on this chronic problem) but examination indicates that the transformed pole is at approximately 30° N and 120° W. Fig. 56A shows the overlay of this oblique transformation on the base projection as performed by the options:

```
+o_proj=mbtfpq +o_lat_p=30 +o_lon_b=-120
```

Fig. 56B shows the transformation with coastlines. An element to note is that the 0° meridian of the transformed system follows the \(\lambda_p\) meridian of the untransformed system. Because the creators of the map wanted to emphasize oceanic regions, the axis were rotated by using \(\lambda_0\). This results in the final options

```
+o_proj=mbtfpq +o_lat_p=30 +o_lon_b=-120 +lon_0=180
```

which results in the map shown in fig. 56C.

Two more examples of transverse pseudocylindrical projections are included here: the Atlantis projection (fig. 57 emphasizes the Atlantic and Arctic Oceans and Close’s map (fig. 58) covers the eastern hemisphere. In the latter map, note that the 20° W and 160° E meridians form a circle.

Use of the general oblique transformation is limited to projections assuming a spherical earth. Oblique or transverse projections on an elliptical earth present complex problem that requires specific analysis of each projection and cannot be applied in a general manner.

---

_A Figure 56: Transverse use of the McBryde-Thomas Flat-Polar Quartic projection: A–oblique transformation on base projection, B–oblique projection with coastlines and C–projection rotated 180° about pole to emphasize oceanic regions._
Figure 57: The Atlantis transverse Mollweide projection, `+proj=ob_tran, +o_proj=moll`, 10° graticule.

Figure 58: Oblique Mollweide projection proposed by Close, `+proj=ob_tran, +o_proj=moll, +o_lat_p=0, +o_lon_p=90, +lon_0=160`. 10° graticule.
References


